Milnor-Witt motivic cohomology of linear algebraic groups and Stiefel varieties

(arXiv:2306.05260)

Keyao Peng

Institut de Mathématiques de Bourgogne

Outline

1. Review of Classic Computations
   - Splitting Cases
   - Non-splitting Cases
   - Integral Coefficient
2. Computations on MW-Motivic Cohomology
   - Homotopy Leray S.S and MW-Cycle Module
   - Non-splitting Cases
   - Applications

Introduction

We aim to compute the cohomology groups of linear Lie groups. For a linear Lie group G, we study the total cohomology group(ring) $H^*(G,R)=\oplus_p H^p(G,R)$ of G itself (instead of classifying space BG) with a coefficient R

Introduction

We aim to compute the cohomology groups of linear Lie groups. For a linear Lie group G, we study the total cohomology group(ring) $H^*(G, R) = \oplus_p H^p(G, R)$ of G itself (instead of classifying space BG) with a coefficient R

Example (linear Lie groups)

- $\mathrm{GL}_n(\mathbb{R}) \cong O(n)$, $\mathrm{SL}_n(\mathbb{R}) \cong SO(n)$
- $\mathrm{GL}_n(\mathbb{C}) \cong U(n)$
- $\mathrm{Sp}_{2n}(\mathbb{C}) \cong Sp(n)$

Leray-Serre Spectral Sequence

The main tool of computation is **Leray-Serre Spectral Sequence**(LS s.s):

Leray-Serre Spectral Sequence

Given a homotopy fiber sequence

$$F \hookrightarrow T \xrightarrow{p} B$$

such that B is simply connected,

Leray-Serre Spectral Sequence

The main tool of computation is **Leray-Serre Spectral Sequence**(LS s.s):

Leray-Serre Spectral Sequence

Given a homotopy fiber sequence

$$F \hookrightarrow T \xrightarrow{p} B$$

such that B is simply connected, then there exists a cohomology Leray-Serre spectral sequence of the form:

$$E_2^{p,q} = H^P(B, H^q(F, R)) \Rightarrow H^{p+q}(T, R)$$

And the differentials satisfy Leibniz rule.

Leray-Serre Spectral Sequence

The main tool of computation is **Leray-Serre Spectral Sequence**(LS s.s):

Leray-Serre Spectral Sequence

Given a homotopy fiber sequence

$$F \hookrightarrow T \xrightarrow{p} B$$

such that B is simply connected, then there exists a cohomology Leray-Serre spectral sequence of the form:

$$E_2^{p,q} = H^P(B, H^q(F, R)) \Rightarrow H^{p+q}(T, R)$$

And the differentials satisfy Leibniz rule.
If we know that $H^q(F, R)$ is free over R, then we can further reduce $H^P(B, H^q(F, R))$ to $H^P(B, R) \otimes_R H^q(F, R)$

Examples

- $SO(n) \hookrightarrow SO(n+1) \xrightarrow{p} \mathbb{R}^{n+1} \setminus \{0\} \cong S^n$

$$E_2^{p,q} = H^P(S^n, H^q(SO(n), R)) \Rightarrow H^{p+q}(SO(n+1), R)$$

Examples

- $SO(n) \hookrightarrow SO(n+1) \xrightarrow{p} \mathbb{R}^{n+1} \setminus \{0\} \cong S^n$

$$E_2^{p,q} = H^P(S^n, H^q(SO(n), R)) \Rightarrow H^{p+q}(SO(n+1), R)$$

- $U(n) \hookrightarrow U(n+1) \xrightarrow{p} \mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1}$

$$E_2^{p,q} = H^P(S^{2n+1}, H^q(U(n), R)) \Rightarrow H^{p+q}(U(n+1), R)$$

Examples

- $SO(n) \hookrightarrow SO(n+1) \xrightarrow{p} \mathbb{R}^{n+1} \setminus \{0\} \cong S^n$

$$E_2^{p,q} = H^P(S^n, H^q(SO(n), R)) \Rightarrow H^{p+q}(SO(n+1), R)$$

- $U(n) \hookrightarrow U(n+1) \xrightarrow{p} \mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1}$

$$E_2^{p,q} = H^P(S^{2n+1}, H^q(U(n), R)) \Rightarrow H^{p+q}(U(n+1), R)$$

- $Sp(n) \hookrightarrow Sp(n+1) \xrightarrow{p} \mathbb{C}^{2n+2} \setminus \{0\} \cong S^{4n+3}$

$$E_2^{p,q} = H^P(S^{4n+3}, H^q(Sp(n), R)) \Rightarrow H^{p+q}(Sp(n+1), R)$$

k-fiber family

These examples can be generalized to the following notion:

Definition

Let a family $\{G_n\}, n \in \mathbb{N}$ of spaces to be a **k-fiber family** for some $k \in \mathbb{N}^*$ if there exists for each G_n a locally trivial fibration(fiber bundle)

$$G_{n-1} \hookrightarrow G_n \xrightarrow{f_n} S^{kn-1},$$

with $G_0 = *$.

k-fiber family

These examples can be generalized to the following notion:

Definition

Let a family $\{G_n\}, n \in \mathbb{N}$ of spaces to be a **k-fiber family** for some $k \in \mathbb{N}^*$ if there exists for each G_n a locally trivial fibration(fiber bundle)

$$G_{n-1} \hookrightarrow G_n \xrightarrow{f_n} S^{kn-1},$$

with $G_0 = *$.

Almost all the fibrations (except some trivial cases) satisfy the condition of LS s.s, thus we have:

$$E_2^{p,q} = H^P(S^{kn-1}, H^q(G_{n-1}, R)) \Rightarrow H^{p+q}(G_n, R)$$

Splitting Cases

If we assume $k > 1$ and $H^q(G_{n-1}, R)$ is free and s.s collapses at E_2 page, we obtain that

$$H^*(G_n, R) \cong H^*(G_{n-1}, R) \otimes_R H^*(S^{kn-1}, R)$$

While $H^*(S^n, R) = R[\alpha_n]_{sq=0}$ is the square-zero graded commutative algebra generated by α_n, the degree $|\alpha_n| = n$.

Splitting Cases

If we assume $k > 1$ and $H^q(G_{n-1}, R)$ is free and s.s collapses at E_2 page, we obtain that

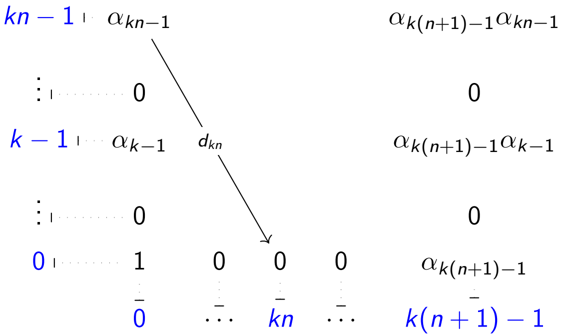
$$H^*(G_n, R) \cong H^*(G_{n-1}, R) \otimes_R H^*(S^{kn-1}, R)$$

While $H^*(S^n, R) = R[\alpha_n]_{sq=0}$ is the square-zero graded commutative algebra generated by α_n, the degree $|\alpha_n| = n$.
Inductively we obtain

$$H^*(G_n, R) \cong R[\alpha_{k-1}, \dots, \alpha_{kn-1}]_{sq=0}$$

This is a free R module, and we will show it makes the s.s of G_{n+1} collapse.

$$E_2^{p,q} = H^P(S^{k(n+1)-1}, H^q(G_n, R)) \Rightarrow H^{p+q}(G_{n+1}, R)$$



Examples

- $\{U(n)\}$ is a 2-fiber family, $U(n) \hookrightarrow U(n+1) \xrightarrow{p} S^{2n+1}$, thus we have:

$$H^*(U(n), R) \cong R[\alpha_1, \dots, \alpha_{2n-1}]_{sq=0}$$

Examples

- $\{U(n)\}$ is a 2-fiber family, $U(n) \hookrightarrow U(n+1) \xrightarrow{p} S^{2n+1}$, thus we have:

$$H^*(U(n), R) \cong R[\alpha_1, \ldots, \alpha_{2n-1}]_{sq=0}$$

- $\{Sp(n)\}$ is a 4-fiber family, $Sp(n) \hookrightarrow Sp(n+1) \xrightarrow{p} S^{4n+3}$, thus we have

$$H^*(Sp(n), R) \cong R[\alpha_3, \ldots, \alpha_{4n-1}]_{sq=0}$$

What about 1-fiber family

Let's consider the simplest example: $SO(2) \cong S^1 \hookrightarrow SO(3) \xrightarrow{p} S^2$

$$E_2^{p,q} = H^P(S^2, H^q(S^1, R)) \Rightarrow H^{p+q}(SO(3), R)$$

What about 1-fiber family

Let's consider the simplest example: $SO(2) \cong S^1 \hookrightarrow SO(3) \xrightarrow{p} S^2$

$$E_2^{p,q} = H^P(S^2, H^q(S^1, R)) \Rightarrow H^{p+q}(SO(3), R)$$

$$\begin{tikzcd}
1 & \alpha_1 \arrow[rrd, "d_2"] & 0 & \alpha_2\alpha_1 \\
0 & 1 & 0 & \alpha_2 \\
 & 0 & 1 & 2
\end{tikzcd}$$

What about 1-fiber family

Let's consider the simplest example: $SO(2) \cong S^1 \hookrightarrow SO(3) \xrightarrow{p} S^2$

$$E_2^{p,q} = H^P(S^2, H^q(S^1, R)) \Rightarrow H^{p+q}(SO(3), R)$$

$$\begin{tikzcd}
1 & \alpha_1 \arrow[drr, "d_2"] & 0 & \alpha_2\alpha_1 \\
0 & 1 & 0 & \alpha_2 \\
 & 0 & 1 & 2
\end{tikzcd}$$

Main Issue

The higher differentials can be non-trivial.

What about 1-fiber family

Let's consider the simplest example: $SO(2) \cong S^1 \hookrightarrow SO(3) \xrightarrow{p} S^2$

$$E_2^{p,q} = H^P(S^2, H^q(S^1, R)) \Rightarrow H^{p+q}(SO(3), R)$$

$$\begin{tikzcd}
1 & \alpha_1 \arrow[rrd, "d_2"] & 0 & \alpha_2\alpha_1 \\
0 & 1 & 0 & \alpha_2 \\
 & 0 & 1 & 2
\end{tikzcd}$$

Main Issue

The higher differentials can be non-trivial.

Idea

d_1 relates to monodromy, The higher differentials relate to holonomy.

Sphere Bundles

Let's consider more general cases of sphere bundle associated to the tangent space of sphere $S(TS^n)$ (where $S(TS^2) \cong SO(3)$), and the fiber sequence: $S^{n-1} \hookrightarrow S(TS^n) \xrightarrow{p} S^n$

Sphere Bundles

Let's consider more general cases of sphere bundle associated to the tangent space of sphere $S(TS^n)$ (where $S(TS^2) \cong SO(3)$), and the fiber sequence: $S^{n-1} \hookrightarrow S(TS^n) \xrightarrow{p} S^n$

$$\begin{tikzcd}
n-1 & \alpha_{n-1} \arrow[ddrr, "d_n"] & 0 & \alpha_n\alpha_{n-1} \\
\vdots & 0 & & 0 \\
0 & 1 & 0 & \alpha_n \\
 & 0 & \cdots & n
\end{tikzcd}$$

We claim $d_n = 1 + (-1)^n$.

Clutching

We can understand the sphere bundle with the "clutching" functions. Consider the northern and southern polar points $1, -1$ and the following open-closed decomposition:

$$\begin{tikzcd}
E \cong S^{n-1} \times (S^{n-1} \times \mathbb{R}) \arrow[r, hook, "\circ" marking] \arrow[d] & S(TS^n) \arrow[d] & S_1^{n-1} \sqcup S_{-1}^{n-1} \arrow[l, hook', "|" marking] \arrow[d] \\
S^n \backslash \{1, -1\} \cong S^{n-1} \times \mathbb{R} \arrow[r, hook, "\circ" marking] & S^n & \{1, -1\} \arrow[l, hook', "|" marking]
\end{tikzcd}$$

Clutching

We can understand the sphere bundle with the "clutching" functions. Consider the northern and southern polar points $1, -1$ and the following open-closed decomposition:

$$\begin{tikzcd}
E \cong S^{n-1} \times (S^{n-1} \times \mathbb{R}) \arrow[r, hook, "\circ" marking] \arrow[d] & S(TS^n) \arrow[d] & S^{n-1}_{1} \sqcup S^{n-1}_{-1} \arrow[l, hook', "|" marking] \arrow[d] \\
S^n \backslash \{1, -1\} \cong S^{n-1} \times \mathbb{R} \arrow[r, hook, "\circ" marking] & S^n & \{1, -1\} \arrow[l, hook', "|" marking]
\end{tikzcd}$$

This induces a Gysin map:

$$\theta : S(TS^n)/E \cong (D^n/S^{n-1}) \wedge (S^{n-1}_{1+} \sqcup S^{n-1}_{-1+}) \to \Sigma E_{+} \cong \Sigma(S^{n-1} \times S^{n-1})_{+}$$

which measures how $S^{n-1}_{1} \sqcup S^{n-1}_{-1}$ is glued along the boundary of E and can give the holonomy data to determine the higher differentials.

Holonomy

When we fix base points $s_1 \in S^{n-1}_1, s_{-1} \in S^{n-1}_{-1}$, and a fiber $S^{n-1} \subset E$, then we can define the (homologic) holonomy:

$$H : S^{n-1} \times \{1, -1\} \xrightarrow{s_1 \oplus s_{-1}} S^{n-1} \wedge (S^{n-1}_{1+} \sqcup S^{n-1}_{-1+}) \xrightarrow{\Sigma^{-1}\theta} E_+ \xrightarrow{pr_2} S^{n-1}$$

And we can show H is given by $(\vec{s} \in S^{n-1}, \epsilon \in \{1, -1\}) \mapsto \epsilon\vec{s}$, thus 1 and $(-1)^n$. This explains why $d_n = 1 + (-1)^n$.

Cohomology of Sphere Bundles

Consequently, we can compute the cohomology groups of $S(TS^n)$ from the s.s:

$$H^i(S(TS^n),R) \cong \begin{cases} R & i = 0, 2n-1 \\ R & n \text{ odd, } i = n-1, n \\ R_{[2]}(\text{2-torsion of } R) & n \text{ even, } i = n-1 \\ R/2 & n \text{ even, } i = n \\ 0 & \text{otherwise} \end{cases}$$

Cohomology of Sphere Bundles

Consequently, we can compute the cohomology groups of $S(TS^n)$ from the s.s:

$$H^i(S(TS^n),R)\cong\begin{cases} R & i=0,2n-1\\ R & n\text{ odd, } i=n-1,n\\ R_{[2]}(\text{2-torsion of } R) & n\text{ even, } i=n-1\\ R/2 & n\text{ even, } i=n\\ 0 & \text{otherwise}\end{cases}$$

In particular, if 2 is invertible in R, we have:

$$H^*(S(TS^n),R)\cong\begin{cases} R[\alpha_{n-1},\alpha_n]_{sq=0} & n\text{ odd}\\ R[\beta_{2n-1}]_{sq=0} & n\text{ even}\end{cases}$$

Stiefel Manifolds(Varieties)

Now we consider even more general cases. For $k \leq n$, we define the Stiefel manifold $V_k(\mathbb{R}^n)$ as "the space of k-rank $k \times n$ matrices".

Stiefel Manifolds(Varieties)

Now we consider even more general cases. For $k \leq n$, we define the Stiefel manifold $V_k(\mathbb{R}^n)$ as "the space of k-rank $k \times n$ matrices".

- $V_n(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \cong O(n)$, $V_{n-1}(\mathbb{R}^n) \cong \mathrm{SL}_n(\mathbb{R}) \cong SO(n)$

Stiefel Manifolds(Varieties)

Now we consider even more general cases. For $k \leq n$, we define the Stiefel manifold $V_k(\mathbb{R}^n)$ as "the space of k-rank $k \times n$ matrices".

- $V_n(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \cong O(n)$, $V_{n-1}(\mathbb{R}^n) \cong \mathrm{SL}_n(\mathbb{R}) \cong SO(n)$
- $V_1(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} \setminus \{0\} \cong S^n$, $V_2(\mathbb{R}^{n+1}) \cong S(TS^n)$

Stiefel Manifolds(Varieties)

Now we consider even more general cases. For $k \leq n$, we define the Stiefel manifold $V_k(\mathbb{R}^n)$ as "the space of k-rank $k \times n$ matrices".

- $V_n(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \cong O(n)$, $V_{n-1}(\mathbb{R}^n) \cong \mathrm{SL}_n(\mathbb{R}) \cong SO(n)$
- $V_1(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} \setminus \{0\} \cong S^n$, $V_2(\mathbb{R}^{n+1}) \cong S(TS^n)$
- For $l < k$, we have the fiber sequences:

$$V_{k-l}(\mathbb{R}^{n-l}) \times \mathbb{R}^{l(k-l)} \cong V_{k-l}(\mathbb{R}^{n-l}) \hookrightarrow V_k(\mathbb{R}^n) \xrightarrow{p} V_l(\mathbb{R}^n)$$

  where p is the restriction to first l rows.

These fiber sequences allow us to make general computations with LS s.s. We first assume 2 is invertible in R.

S.S of Stiefel Manifolds

- When $n-k$ is even, we consider the fiber sequence and s.s:

$$V_1(\mathbb{R}^{n-k+1}) \cong S^{n-k} \hookrightarrow V_k(\mathbb{R}^n) \xrightarrow{p} V_{k-1}(\mathbb{R}^n)$$

$$E_2^{p,q} \cong H^p(V_{k-1}(\mathbb{R}^n), R) \otimes H^q(S^{n-k}, R) \Rightarrow H^{p+q}(V_k(\mathbb{R}^n), R)$$

S.S of Stiefel Manifolds

- When $n-k$ is even, we consider the fiber sequence and s.s:

$$V_1(\mathbb{R}^{n-k+1}) \cong S^{n-k} \hookrightarrow V_k(\mathbb{R}^n) \xrightarrow{p} V_{k-1}(\mathbb{R}^n)$$

$$E_2^{p,q} \cong H^p(V_{k-1}(\mathbb{R}^n), R) \otimes H^q(S^{n-k}, R) \Rightarrow H^{p+q}(V_k(\mathbb{R}^n), R)$$

- When $n-k$ is odd, we consider the fiber sequence and s.s:

$$V_2(\mathbb{R}^{n-k+1}) \cong S(TS^{n-k+1}) \hookrightarrow V_k(\mathbb{R}^n) \xrightarrow{p} V_{k-2}(\mathbb{R}^n)$$

$$E_2^{p,q} \cong H^p(V_{k-2}(\mathbb{R}^n), R) \otimes H^q(S(TS^{n-k+1}), R) \Rightarrow H^{p+q}(V_k(\mathbb{R}^n), R)$$

With similar augments as before, we can show the s.s in both cases collapse at E_2 page.

Cohomology of Stiefel Manifolds

With the same augments and induction, we have that

$$H^*(V_k(\mathbb{R}^n), R) \cong \begin{cases} H^*(V_{k-1}(\mathbb{R}^n), R) \otimes H^*(S^{n-k}, R) & n-k \text{ even} \\ H^*(V_{k-2}(\mathbb{R}^n), R) \otimes H^*(S(TS^{n-k+1}), R) & n-k \text{ odd} \end{cases}$$

And we can present the total cohomology rings as free square-zero algebras.

Cohomology of Stiefel Manifolds

With the same augments and induction, we have that

$$H^*(V_k(\mathbb{R}^n),R)\cong\begin{cases}H^*(V_{k-1}(\mathbb{R}^n),R)\otimes H^*(S^{n-k},R) & n-k \text{ even}\\ H^*(V_{k-2}(\mathbb{R}^n),R)\otimes H^*(S(TS^{n-k+1}),R) & n-k \text{ odd}\end{cases}$$

And we can present the total cohomology rings as free square-zero algebras.

Corollary

$$H^*(SO(n),R)\cong H^*(V_{n-1}(\mathbb{R}^n),R)\cong\begin{cases}R[\alpha_{n-1},\beta_{4i-1}|0<2i<n]_{sq=0} & n \text{ even}\\ R[\beta_{4i-1}|0<2i<n]_{sq=0} & n \text{ odd}\end{cases}$$

$$H^*(O(n),R)\cong H^*(V_n(\mathbb{R}^n),R)\cong\begin{cases}R[\alpha_0,\alpha_{n-1},\beta_{4i-1}|0<2i<n]_{sq=0} & n \text{ even}\\ R[\alpha_0,\beta_{4i-1}|0<2i<n]_{sq=0} & n \text{ odd}\end{cases}$$

Thom-Gysin Sequence

Now we consider the cases that 2 is non-invertible, in particular the integral coefficient $\mathbb{Z}$. Again with the following fibration and s.s

$$V_1(\mathbb{R}^{n-k}) \cong S^{n-k-1} \hookrightarrow V_{k+1}(\mathbb{R}^n) \xrightarrow{p} V_k(\mathbb{R}^n)$$

$$E_2^{p,q} \cong H^p(V_k(\mathbb{R}^n), \mathbb{Z}) \otimes H^q(S^{n-k-1}, \mathbb{Z}) \Rightarrow H^{p+q}(V_{k+1}(\mathbb{R}^n), \mathbb{Z})$$

Thom-Gysin Sequence

Now we consider the cases that 2 is non-invertible, in particular the integral coefficient $\mathbb{Z}$. Again with the following fibration and s.s

$$V_1(\mathbb{R}^{n-k}) \cong S^{n-k-1} \hookrightarrow V_{k+1}(\mathbb{R}^n) \xrightarrow{p} V_k(\mathbb{R}^n)$$

$$E_2^{p,q} \cong H^p(V_k(\mathbb{R}^n), \mathbb{Z}) \otimes H^q(S^{n-k-1}, \mathbb{Z}) \Rightarrow H^{p+q}(V_{k+1}(\mathbb{R}^n), \mathbb{Z})$$

Notice that the E_2 page now has only two non-trivial rows and possible non-trivial differentials d_{n-k}. Because of Leibniz rule, these differentials are just cupping with the class $e = d_{n-k}(\theta_{n-k-1})$ where $\theta_{n-k-1} \in H^{n-k-1}(S^{n-k-1})$ is a generator.

Thom-Gysin Sequence

Now we consider the cases that 2 is non-invertible, in particular the integral coefficient $\mathbb{Z}$. Again with the following fibration and s.s

$$V_1(\mathbb{R}^{n-k}) \cong S^{n-k-1} \hookrightarrow V_{k+1}(\mathbb{R}^n) \xrightarrow{p} V_k(\mathbb{R}^n)$$

$$E_2^{p,q} \cong H^p(V_k(\mathbb{R}^n), \mathbb{Z}) \otimes H^q(S^{n-k-1}, \mathbb{Z}) \Rightarrow H^{p+q}(V_{k+1}(\mathbb{R}^n), \mathbb{Z})$$

Notice that the E_2 page now has only two non-trivial rows and possible non-trivial differentials d_{n-k}. Because of Leibniz rule, these differentials are just cupping with the class $e = d_{n-k}(\theta_{n-k-1})$ where $\theta_{n-k-1} \in H^{n-k-1}(S^{n-k-1})$ is a generator.

Therefore, the degeneration of s.s gives the Thom-Gysin long exact sequence:

$$H^{i-n+k}(V_k(\mathbb{R}^n)) \xrightarrow{\cup e} H^i(V_k(\mathbb{R}^n)) \xrightarrow{p^*} H^i(V_{k+1}(\mathbb{R}^n)) \to H^{i-n+k+1}(V_k(\mathbb{R}^n)) \xrightarrow{\cup e}$$

Euler Class

The e is actually the Euler class $e(f_{n,k})$ corresponding the bundle $V_{k+1}(\mathbb{R}^n) \xrightarrow{f_{n,k}} V_k(\mathbb{R}^n)$. In particular, by previous discussion, $e(f_{n,1})$ is 0 for n even, $2\theta_{n-1} \in H^{n-1}(S^{n-1})$ for n odd.

Euler Class

The e is actually the Euler class $e(f_{n,k})$ corresponding the bundle $V_{k+1}(\mathbb{R}^n) \xrightarrow{f_{n,k}} V_k(\mathbb{R}^n)$. In particular, by previous discussion, $e(f_{n,1})$ is 0 for n even, $2\theta_{n-1} \in H^{n-1}(S^{n-1})$ for n odd. And in fact, because of following pullback of bundle, all $e(f_{n,k})$ behave like this.

$$\begin{tikzcd}
S^{n-k-1} \arrow[r] \arrow[d, equal] & V_2(\mathbb{R}^{n-k+1}) \arrow[r, "f_{n-k+1,1}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S^{n-k} \arrow[d, "g"] \\
S^{n-k-1} \arrow[r] & V_{k+1}(\mathbb{R}^n) \arrow[r, "f_{n,k}"] \arrow[d] & V_k(\mathbb{R}^n) \arrow[d] \\
 & V_{k-1}(\mathbb{R}^n) \arrow[r, equal] & V_{k-1}(\mathbb{R}^n)
\end{tikzcd}$$

Integral Cohomology

Let $H(V_{k,n}) := H^*(V_k(\mathbb{R}^n), \mathbb{Z})$, combining the Thom-Gysin sequence and Euler class, we can have the inductive computations:

$$H(V_{k+1,n}) \cong \begin{cases} H(V_{k,n}) \otimes H(S^{n-k}, \mathbb{Z}) \cong & \\ H(V_{k,n}) \oplus H(V_{k,n})\alpha_{n-k} & n-k \text{ even} \\ H(V_{k-1,n}) \oplus (H(V_{k-1,n})/2)\alpha_{n-k} \oplus & \\ H(V_{k-1,n})_{[2]}\alpha_{n-k-1} \oplus H(V_{k-1,n})\beta_{2(n-k)-1} & n-k \text{ odd} \end{cases}$$

Integral Cohomology

Let $H(V_{k,n}) := H^*(V_k(\mathbb{R}^n), \mathbb{Z})$, combining the Thom-Gysin sequence and Euler class, we can have the inductive computations:

$$H(V_{k+1,n}) \cong \begin{cases} H(V_{k,n}) \otimes H(S^{n-k}, \mathbb{Z}) \cong & \\ H(V_{k,n}) \oplus H(V_{k,n})\alpha_{n-k} & n-k \text{ even} \\ H(V_{k-1,n}) \oplus (H(V_{k-1,n})/2)\alpha_{n-k} \oplus & \\ H(V_{k-1,n})_{[2]}\alpha_{n-k-1} \oplus H(V_{k-1,n})\beta_{2(n-k)-1} & n-k \text{ odd} \end{cases}$$

Therefore we can compute the integral cohomology group inductively, as a module it is a direct sum of $\mathbb{Z}$ and $\mathbb{Z}/2$. But the ring structure is more complicated.

Outline

.

1. Review of Classic Computations
   - Splitting Cases
   - Non-splitting Cases
   - Integral Coefficient

2. Computations on MW-Motivic Cohomology
   - Homotopy Leray S.S and MW-Cycle Module
   - Non-splitting Cases
   - Applications

Dictionary between Classic and $\mathbb{A}^1$ Homotopy Theory

To copy the proof of the classic case, we first summarize some correspondence between classic and $\mathbb{A}^1$ homotopy theory:

Classic	$\xleftarrow{real\ \ realization} \mathbb{A}^1$
$\mathbb{R}$	$\mathbb{A}^1$
$\mathbb{R}^* \cong \mathbb{Z}/2$	$\mathbb{G}_m$
S^n	$\mathbb{A}^{n+1} \setminus \{0\} \cong S^n \wedge \mathbb{G}_m^{\wedge n+1}$
$V_k(\mathbb{R}^n)$	$V_k(\mathbb{A}^n)$
$O(n), SO(n), U(n)(Sp(n))$	$\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$
Abelian group	MW-cycle module over field K
Local system(D-module) over X	MW-cycle module over scheme X
$\mathbb{Z}, \mathbb{Z}[1/2]$	$\mathrm{K}_*^{\mathrm{MW}}, \mathrm{K}_*^{\mathrm{MW}}[\eta^{-1}] \cong W_*$
$H^p(X, \mathbb{Z})$	$H^p(X, \mathrm{K}_*^{\mathrm{MW}})$ or $\mathrm{H}_{\mathrm{MW}}^{p+*,*}(X, \mathbb{Z})$
$H^*(S^n, \mathbb{Z}) \cong \mathbb{Z}[\alpha_n]_{sq=0}$	$H^*(\mathbb{A}^{n+1} \setminus \{0\}, \mathrm{K}_*^{\mathrm{MW}}) \cong \mathrm{K}_{\{*\}}^{\mathrm{MW}}(K)[\alpha_{n,\{n+1\}}]_{sq=0}$

Homotopy Leray Spectral Sequence

The key ingredient is LS s.s in $\mathbb{A}^1$ setting, which was introduced in (arXiv:1812.09574):

Homotopy Leray S.S (Asok-Déglise-Nagel)

Let $f : T \to B$ be a morphism of schemes, and let $\mathbb{E} \in \mathcal{SH}(T)$ be a ring spectrum over T, we have a spectral sequence of the form:

$$E_2^{p,q}(f, \mathbb{E}) = H^p(B, H_\delta^q f_* \mathbb{E}) \Rightarrow H^{p+q}(T, \mathbb{E})$$

where the differentials d_r satisfy the usual Leibniz rule.

Homotopy Leray Spectral Sequence

The key ingredient is LS s.s in $\mathbb{A}^1$ setting, which was introduced in (arXiv:1812.09574):

Homotopy Leray S.S (Asok-Déglise-Nagel)

Let $f : T \to B$ be a morphism of schemes, and let $\mathbb{E} \in \mathcal{SH}(T)$ be a ring spectrum over T, we have a spectral sequence of the form:

$$E_2^{p,q}(f, \mathbb{E}) = H^p(B, H_\delta^q f_* \mathbb{E}) \Rightarrow H^{p+q}(T, \mathbb{E})$$

where the differentials d_r satisfy the usual Leibniz rule.

Issue

We don't have a good notion of "simply connected". The "derived push forward" $H_\delta^q f_* \mathbb{E}$(i.e. $R^q f_* \mathbb{E}$) can be a "local system" instead of a "constant coefficient".

Monodromy of MW-cycle module

We can explicitly study the "monodromy" of the push forward MW-cycle module.
Let's consider a simplest example: Given a fiber bundle $T \xrightarrow{p} B$, and we have$\{U_i\}$ is a (Zariski) open covering of B, such that we have trivialization $\chi_i : U_i \times_B T \xrightarrow{\cong} U_i \times F$.

Monodromy of MW-cycle module

We can explicitly study the "monodromy" of the push forward MW-cycle module.
Let's consider a simplest example: Given a fiber bundle $T \xrightarrow{p} B$, and we have$\{U_i\}$ is a (Zariski) open covering of B, such that we have trivialization $\chi_i : U_i \times_B T \xrightarrow{\cong} U_i \times F$.
Now on the intersection $U_i \cap U_j$, different trivializations χ_i and χ_j give rise to a transform map $g_{ij} : (U_i \cap U_j) \times F \xrightarrow{\cong} (U_i \cap U_j) \times F$. $\{g_{ij}\}$ defines a $Aut(F)$-torsor and a class in $H^1(B, Aut(F))$, which induces a class $\tau \in H^1(B, Aut(H^*(F, \mathbb{E})))$.

Monodromy of MW-cycle module

We can explicitly study the "monodromy" of the push forward MW-cycle module.
Let's consider a simplest example: Given a fiber bundle $T \xrightarrow{p} B$, and we have$\{U_i\}$ is a (Zariski) open covering of B, such that we have trivialization $\chi_i : U_i \times_B T \xrightarrow{\cong} U_i \times F$.
Now on the intersection $U_i \cap U_j$, different trivializations χ_i and χ_j give rise to a transform map $g_{ij} : (U_i \cap U_j) \times F \xrightarrow{\cong} (U_i \cap U_j) \times F$. $\{g_{ij}\}$ defines a $Aut(F)$-torsor and a class in $H^1(B, Aut(F))$, which induces a class $\tau \in H^1(B, Aut(H^*(F, \mathbb{E})))$.
For F and $\mathbb{E}$ good enough (e.g.$H^*(F, \mathbb{E})$ is free), this τ gives the monodromy of $H^q_\delta f_* \mathbb{E}$. Instead of use "simply connectedness", we can show directly τ is trivial.

Example of $V_2(\mathbb{A}^n)$

$$\mathbb{A}^{n-1} \setminus \{0\} \hookrightarrow V_2(\mathbb{A}^n) \xrightarrow{p} \mathbb{A}^n \setminus \{0\}$$

$$p : \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \mapsto [x_1, \cdots, x_n]$$

Take the opens $U_i = \{x_i \neq 0\} \subset \mathbb{A}^n \setminus \{0\}$ as covering. Then the trivialization $\chi_i : p^{-1}(U_i) \to U_i \times \mathbb{A}^{n-1} \setminus \{0\}$ is given by

Example of $V_2(\mathbb{A}^n)$

$$\mathbb{A}^{n-1} \setminus \{0\} \hookrightarrow V_2(\mathbb{A}^n) \xrightarrow{p} \mathbb{A}^n \setminus \{0\}$$

$$p : \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \mapsto [x_1, \cdots, x_n]$$

Take the opens $U_i = \{x_i \neq 0\} \subset \mathbb{A}^n \setminus \{0\}$ as covering. Then the trivialization $\chi_i : p^{-1}(U_i) \to U_i \times \mathbb{A}^{n-1} \setminus \{0\}$ is given by

$$\chi_i : \begin{bmatrix} x_1 & \cdots & x_i* & \cdots & x_n \\ y_1 & \cdots & y_i & \cdots & y_n \end{bmatrix} \mapsto [x_1, \cdots, x_n], [y_1 - y_i x_1/x_i, \cdots, y_n - y_i x_n/x_i]$$

The transform map g_{ij} satisfies:

$$g_{ij} : [y_1 - y_i x_1/x_i, \cdots, y_n - y_i x_n/x_i] \mapsto [y_1 - y_j x_1/x_j, \cdots, y_n - y_j x_n/x_j]$$

The transform map g_{ij} satisfies:

$$g_{ij}:[y_1-y_ix_1/x_i,\cdots,y_n-y_ix_n/x_i]\mapsto[y_1-y_jx_1/x_j,\cdots,y_n-y_jx_n/x_j]$$

If you solve the equations, you will find up to some permutation, g_{ij} is given by matrix:

$$\begin{bmatrix}
1 & & & & & & & & 0 \\
 & \ddots & & & & & & & \\
 & & 1 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & \ddots & & & & \\
0 & & & & & 1 & & & \\
-x_1/x_j & \cdots & -x_{i-1}/x_j & -x_{i+1}/x_j & \cdots & -x_{j-1}/x_j & -x_i/x_j & \cdots & -x_n/x_j \\
 & & & & & & & \ddots &
\end{bmatrix}$$

Up to $\mathbb{A}^1$ homotopy, $g_{ij} \sim diag((-1)^{i-j}x_i/x_j, 1, \ldots, 1)$. Then we notice that $diag((-1)^{i-j}x_i/x_j, 1, \ldots, 1) = diag((-1)^i x_i, 1, \ldots, 1) diag((-1)^j x_j, 1, \ldots, 1)^{-1} = h_i h_j^{-1}$.

Up to $\mathbb{A}^1$ homotopy, $g_{ij} \sim diag((-1)^{i-j}x_i/x_j, 1, \ldots, 1)$.
Then we notice that $diag((-1)^{i-j}x_i/x_j, 1, \ldots, 1) = diag((-1)^i x_i, 1, \ldots, 1) diag((-1)^j x_j, 1, \ldots, 1)^{-1} = h_i h_j^{-1}$.
This means the induced class is trivial, i.e.

$$[g_{ij}] = 0 \in H^1(\mathbb{A}^n \setminus \{0\}, Aut(H^*(\mathbb{A}^{n-1} \setminus \{0\})))$$

Consequently, in this case, taking $\mathbb{E} = \mathrm{K}^{\mathrm{MW}}$ we can reduce the s.s as

$$E_2^{p,q} = H^p(\mathbb{A}^n \setminus \{0\}, H^q(\mathbb{A}^{n-1} \setminus \{0\}, \mathrm{K}^{\mathrm{MW}})) \Rightarrow H^{p+q}(V_2(\mathbb{A}^n), \mathrm{K}^{\mathrm{MW}})$$

Splitting Cases

Similarly, we can show the monodromy is trivial for any such fiber sequence, etc:

$$V_{k-l}(\mathbb{A}^{n-l}) \hookrightarrow V_k(\mathbb{A}^n) \xrightarrow{p} V_l(\mathbb{A}^n)$$

This allows us to copy the proof for k-fiber family, where $k > 1$.

Splitting Cases

Similarly, we can show the monodromy is trivial for any such fiber sequence, etc:

$$V_{k-l}(\mathbb{A}^{n-l}) \hookrightarrow V_k(\mathbb{A}^n) \xrightarrow{p} V_l(\mathbb{A}^n)$$

This allows us to copy the proof for k-fiber family, where $k > 1$.

Example

$\{\mathrm{Sp}_{2n}\}$ is a 2-fiber family in $\mathbb{A}^1$ homotopy theory. Therefore we have:

$$H^*(\mathrm{Sp}_{2n}, \mathrm{K}_*^{\mathrm{MW}}) \cong \mathrm{K}_{\{*\}}^{\mathrm{MW}}(K)[\alpha_{1,\{2\}}, \alpha_{3,\{4\}}, \dots, \alpha_{2n-1,\{2n\}}]_{sq=0}$$

Remark

We can also get the same result for $\mathrm{H}_{\mathrm{MW}}^{*,*}(\mathrm{Sp}_{2n}, \mathbb{Z})$ by using Sp-orientation and Borel classes.

Sp-Orientation and Borel Classes

For a generalized cohomology theory $E \in \mathcal{SH}(K)$, a Sp-orientation is simply a morphism $\tau : \mathrm{MSp} \to \mathbb{E}$.
For Sp-oriented theories, we have the projective bundle formula and splitting principle, which allows us to define Borel classes $b_j \in \mathbb{E}^{4j,2j}(\mathrm{BSp}_{2n})$ for $1 \leq j \leq n$, satisfying $\mathbb{E}^{*,*}(B\mathrm{Sp}_{2n}) \cong \mathbb{E}^{*,*}(K)[[b_1, \ldots, b_n]]$.
In particular, the Milnor-Witt motivic cohomology H_{MW} is Sp-oriented.

Sp-Orientation and Borel Classes

For a generalized cohomology theory $E \in \mathcal{SH}(K)$, a Sp-orientation is simply a morphism $\tau : \mathrm{MSp} \to \mathbb{E}$.
For Sp-oriented theories, we have the projective bundle formula and splitting principle, which allows us to define Borel classes $b_j \in \mathbb{E}^{4j,2j}(\mathrm{BSp}_{2n})$ for $1 \leq j \leq n$, satisfying $\mathbb{E}^{*,*}(B\mathrm{Sp}_{2n}) \cong \mathbb{E}^{*,*}(K)[[b_1, \ldots, b_n]]$.
In particular, the Milnor-Witt motivic cohomology H_{MW} is Sp-oriented.
Now using loop spaces, we can define a new series of classes $\beta(n)_j \in \mathbb{E}^{4j-1,2j}(\mathrm{Sp}_{2n})$:

$$\beta(n)_j : \mathrm{Sp}_{2n} \cong \Omega B\mathrm{Sp}_{2n} \xrightarrow{\Omega b_j} \Sigma^{4j-1,2j}\mathbb{E},$$

which are compatible between different n.

Sp-oriented cohomology of Sp_{2n}

We compute $\mathbb{E}^{*,*}(\mathrm{Sp}_{2n})$ via fiber sequence

$$\mathrm{Sp}_{2n-2} \hookrightarrow \mathrm{Sp}_{2n} \xrightarrow{f} \mathbb{A}^{2n} \setminus \{0\}$$

$$E_2^{p,q} = H^P(\mathbb{A}^{2n} \setminus \{0\}, \mathbb{E}^{q,*}(\mathrm{Sp}_{2n-2})) \Rightarrow \mathbb{E}^{p+q,*}(\mathrm{Sp}_{2n})$$

Sp-oriented cohomology of Sp_{2n}

We compute $\mathbb{E}^{*,*}(\mathrm{Sp}_{2n})$ via fiber sequence

$$\mathrm{Sp}_{2n-2} \hookrightarrow \mathrm{Sp}_{2n} \xrightarrow{f} \mathbb{A}^{2n} \setminus \{0\}$$

$$E_2^{p,q} = H^P(\mathbb{A}^{2n} \setminus \{0\}, \mathbb{E}^{q,*}(\mathrm{Sp}_{2n-2})) \Rightarrow \mathbb{E}^{p+q,*}(\mathrm{Sp}_{2n})$$

Then the classes $\beta(n)_j \in \mathbb{E}^{4j-1,2j}(\mathrm{Sp}_{2n})$ a priori give generators of $E_2^{p,q}$ which with force s.s collapses at E_2 page.

Theorem

For Sp-oriented theory $\mathbb{E}$, in particular H_{MW}, we have the isomorphism of graded module:

$$\mathbb{E}^{*,*}(\mathrm{Sp}_{2n}) \cong \mathbb{E}^{*,*}(K)[\beta(n)_1, \ldots, \beta(n)_n]_{sq=0}$$

Clutching of $V_2(\mathbb{A}^n)$

GL_n is 1-fiber family as in classic cases, thus we also need the same consideration on $V_2(\mathbb{A}^n)$, we have also the open-closed decomposition:

$$\begin{tikzcd}
E \cong \mathbb{A}^{n-1} \setminus \{0\} \times (\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^1) \arrow[r, hook, "\circ" marking] \arrow[d] & V_2(\mathbb{A}^n) \arrow[d] & \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \arrow[l, hook', "|" marking] \arrow[d] \\
\{x_n \neq 0\} \cong \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^1 \arrow[r, hook, "\circ" marking] & \mathbb{A}^n \setminus \{0\} & \{x_n = 0\} \cong \mathbb{G}_m \arrow[l, hook', "|" marking]
\end{tikzcd}$$

And the Gysin map:

$$\theta : (\mathbb{A}^{n-1}/\mathbb{A}^{n-1}\setminus\{0\}) \wedge (\mathbb{G}_m \times \mathbb{A}^{n-1}\setminus\{0\})_+ \to \Sigma(\mathbb{A}^{n-1}\setminus\{0\} \times \mathbb{A}^{n-1}\setminus\{0\})_+$$

Algebraic Holonomy

We can explicitly write a matrix for the "parallel transport"

$$\Sigma^{-1}\theta : \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \to \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\}$$

$$(\vec{x}, x_n, \vec{y}) \mapsto (\vec{x}, \vec{y}(\mathbf{I} + (x_n - 1)\vec{x}^T("\frac{\vec{x}}{|\vec{x}|^2}")))$$

($\frac{\vec{x}}{|\vec{x}|^2}$ is not well-defined algebraically. But we have something $\vec{w}$ to simulate it, such that $(\vec{x} \cdot \vec{w} = 1)$)

Algebraic Holonomy

We can explicitly write a matrix for the "parallel transport"

$$\Sigma^{-1}\theta : \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \to \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\}$$

$$(\vec{x}, x_n, \vec{y}) \mapsto (\vec{x}, \vec{y}(\mathbf{I} + (x_n - 1)\vec{x}^T('' \frac{\vec{x}}{|\vec{x}|^2}'')))$$

($\frac{\vec{x}}{|\vec{x}|^2}$ is not well-defined algebraically. But we have something $\vec{w}$ to simulate it, such that $(\vec{x} \cdot \vec{w} = 1)$)
And we chose the base point by diagonal map, thus the holonomy

$$H : \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{G}_m \xrightarrow{d} \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \xrightarrow{pr_2 \circ \Sigma^{-1}\theta} \mathbb{A}^{n-1} \setminus \{0\}$$

is defined by $(\vec{x}, x_n) \mapsto (\vec{x}, x_n, \vec{x}) \mapsto x_n\vec{x}$

Motivic Decomposition of $V_2\mathbb{A}^n$

In $\mathbb{A}^1$ homotopy theory, η plays the role of 2.

Lemma (Morel-Sawant)

In $D_{\mathbb{A}^1}(K)$, H is given by $n_\epsilon\eta$. i.e. it is 0 if n is even, and equal to η if n is odd.

Motivic Decomposition of $V_2\mathbb{A}^n$

In $\mathbb{A}^1$ homotopy theory, η plays the role of 2.

Lemma (Morel-Sawant)

In $D_{\mathbb{A}^1}(K)$, H is given by $n_\epsilon \eta$. i.e. it is 0 if n is even, and equal to η if n is odd.

Combining with the cofiber sequence comes from open-closed decomposition:

Corollary

The following motivic decompositions hold in $\widetilde{\mathrm{DM}}(K)$:

$$\widetilde{\mathrm{M}}(V_2(\mathbb{A}^{2k})) \cong \widetilde{\mathrm{M}}(\mathbb{A}^{2k} \setminus \{0\}) \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2k-1} \setminus \{0\})$$

$$\widetilde{\mathrm{M}}(V_2(\mathbb{A}^{2k+1})) \cong \widetilde{\mathrm{M}}(K) \oplus C_\eta(K)(2k)[4k] \oplus \widetilde{\mathrm{M}}(K)(4k+1)[8k],$$

where $C_\eta(K)$ is the cone of $\widetilde{\mathrm{M}}(K) \xrightarrow{\eta} \widetilde{\mathrm{M}}(K)(1)[1]$

η-inverted MW-motivic cohomology of Stiefel Varieties

If we invert η, the coefficient $\mathrm{K}_*^{\mathrm{MW}}[\eta^{-1}] = W_*$ only matter for $W_0 = W$. Then as in classic, we can deduce the cohomology of $V_2(\mathbb{A}^n)$:

$$\mathrm{H}_\eta^*(V_2(\mathbb{A}^n)) := H^*(V_2(\mathbb{A}^n), W) \cong \begin{cases} W(K)[\alpha_{n-2}, \alpha_{n-1}]_{sq=0} & n \text{ even} \\ W(K)[\beta_{2n-3}]_{sq=0} & n \text{ odd} \end{cases}$$

η-inverted MW-motivic cohomology of Stiefel Varieties

If we invert η, the coefficient $\mathrm{K}_*^{\mathrm{MW}}[\eta^{-1}] = W_*$ only matter for $W_0 = W$. Then as in classic, we can deduce the cohomology of $V_2(\mathbb{A}^n)$:

$$\mathrm{H}_\eta^*(V_2(\mathbb{A}^n)) := H^*(V_2(\mathbb{A}^n), W) \cong \begin{cases} W(K)[\alpha_{n-2}, \alpha_{n-1}]_{sq=0} & n \text{ even} \\ W(K)[\beta_{2n-3}]_{sq=0} & n \text{ odd} \end{cases}$$

At this moment, we can fully copy the classic proof and get our results:

$$\mathrm{H}_\eta^*(V_k(\mathbb{A}^n)) \cong \begin{cases} \mathrm{H}_\eta^*(V_{k-1}(\mathbb{A}^n)) \otimes_{\mathrm{H}_\eta^*(S)} \mathrm{H}_\eta^*(\mathbb{A}^{n-k+1} \setminus \{0\}) & \text{if } n-k \text{ is even.} \\ \mathrm{H}_\eta^*(V_{k-2}(\mathbb{A}^n)) \otimes_{\mathrm{H}_\eta^*(S)} \mathrm{H}_\eta^*(V_2(\mathbb{A}^{n-k+2})) & \text{if } n-k \text{ is odd.} \end{cases}$$

Theorem

We have an isomorphism of graded modules

$$\mathrm{H}^*_\eta(V_k(\mathbb{A}^n)) \cong W(K)[G]_{sq=0}$$

where the generators G are given by

$$G = \{n \text{ even}, \alpha_{n-1} \in \mathrm{H}^{n-1}_\eta\} \sqcup \{\beta_{4j-1} \in \mathrm{H}^{4j-1}_\eta | n-k < 2j < n\}$$

$$\sqcup \{n-k \text{ even}, \gamma_{n-k} \in \mathrm{H}^{n-k}_\eta\}$$

Corollary

$$\mathrm{H}^*_\eta(\mathrm{SL}_n) \cong \mathrm{H}^*_\eta(V_{n-1}(\mathbb{A}^n)) \cong \begin{cases} W(K)[\alpha_{n-1}, \beta_{4i-1}|0 < 2i < n]_{sq=0} & n \text{ even} \\ W(K)[\beta_{4i-1}|0 < 2i < n]_{sq=0} & n \text{ odd} \end{cases}$$

Integral MW-motivic cohomology of Stiefel Varieties

Similarly, we can use Thom-Gysin sequence and Euler class to compute integral MW-motivic cohomology. The trivial "monodromy" also indicate that the Euler class in H_{MW} is well-defined for bundle $f_{n,k} : V_{k+1}(\mathbb{A}^n) \to V_k(\mathbb{A}^n)$ (i.e it is H_{MW}-oriented).

$$\xrightarrow{\cup e(f_{n,k})} \mathrm{H}_{\mathrm{MW}}^{i,j}(V_k(\mathbb{A}^n)) \to \mathrm{H}_{\mathrm{MW}}^{i,j}(V_{k+1}(\mathbb{A}^n)) \to \mathrm{H}_{\mathrm{MW}}^{i-(2(n-k)-1),j-(n-k)}(V_k(\mathbb{A}^n)) \xrightarrow{\cup e(}$$

With the same consideration as before, we can show the Euler class $e(f_{n,k})$ is $(n-k)_{\epsilon}\eta\alpha_{n-k}$. And we can compute the integral cohomology inductively, just replacing 2 with η.

Let $\mathrm{H}_{\mathrm{MW}}(V_{k,n}) := \mathrm{H}^{*,*}_{\mathrm{MW}}(V_k(\mathbb{A}^n), \mathbb{Z})$, combining the Thom-Gysin sequence and Euler class as in topological cases, we get the similar inductive computations:

$$\mathrm{H}_{\mathrm{MW}}(V_{k+1,n}) \cong \begin{cases} \mathrm{H}_{\mathrm{MW}}(V_{k,n}) \otimes \mathrm{H}_{\mathrm{MW}}(\mathbb{A}^{n-k+1} \setminus \{0\}, \mathbb{Z}) \cong & \\ \mathrm{H}_{\mathrm{MW}}(V_{k,n}) \oplus \mathrm{H}_{\mathrm{MW}}(V_{k,n})\alpha_{n-k} & n-k \text{ even} \\ \mathrm{H}_{\mathrm{MW}}(V_{k-1,n}) \oplus (\mathrm{H}_{\mathrm{MW}}(V_{k-1,n})/\eta)\alpha_{n-k} \oplus & \\ \mathrm{H}_{\mathrm{MW}}(V_{k-1,n})_{[\eta]}\alpha_{n-k-1} \oplus \mathrm{H}_{\mathrm{MW}}(V_{k-1,n})\beta_{2(n-k)-1} & n-k \text{ odd} \end{cases}$$

Let $\mathrm{H}_{\mathrm{MW}}(V_{k,n}) := \mathrm{H}^{*,*}_{\mathrm{MW}}(V_k(\mathbb{A}^n), \mathbb{Z})$, combining the Thom-Gysin sequence and Euler class as in topological cases, we get the similar inductive computations:

$$\mathrm{H}_{\mathrm{MW}}(V_{k+1,n}) \cong \begin{cases} \mathrm{H}_{\mathrm{MW}}(V_{k,n}) \otimes \mathrm{H}_{\mathrm{MW}}(\mathbb{A}^{n-k+1} \setminus \{0\}, \mathbb{Z}) \cong \\ \mathrm{H}_{\mathrm{MW}}(V_{k,n}) \oplus \mathrm{H}_{\mathrm{MW}}(V_{k,n})\alpha_{n-k} & n-k \text{ even} \\ \mathrm{H}_{\mathrm{MW}}(V_{k-1,n}) \oplus (\mathrm{H}_{\mathrm{MW}}(V_{k-1,n})/\eta)\alpha_{n-k} \oplus \\ \mathrm{H}_{\mathrm{MW}}(V_{k-1,n})_{[\eta]}\alpha_{n-k-1} \oplus \mathrm{H}_{\mathrm{MW}}(V_{k-1,n})\beta_{2(n-k)-1} & n-k \text{ odd} \end{cases}$$

Again we can compute the integral cohomology group inductively, as a module it is a direct sum of $\mathrm{H}^{*,*}_{\mathrm{MW}}(K)$, $\mathrm{H}^{*,*}_{\mathrm{MW}}(K)/\eta = \mathrm{H}^{*,*}_{\mathrm{M}}(K)$ and $\mathrm{H}^{*,*}_{\mathrm{MW}}(K)_{[\eta]}$.

Applications

Knowing the cohomology groups + some cofiber sequences, we can get the MW-motivic decompositions.

For Sp_{2n}, we can get the decomposition in $\widetilde{\mathrm{DM}}(K)$:

Decomposition of Sp_{2n}

$$\widetilde{\mathrm{M}}(\mathrm{Sp}_{2n}) \cong \widetilde{\mathrm{M}}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2n-2} \setminus \{0\}) \otimes \ldots \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2} \setminus \{0\}) \in \widetilde{\mathrm{DM}}(K)$$

Applications

And for SL_n and GL_n we consider the decompositions in $\widetilde{\mathrm{DM}}(K)[\eta^{-1}]$,

$$\mathbf{HS}_{2k} := \widetilde{\mathrm{M}}_\eta(\mathbb{A}^{2k} \setminus \{0\}) \cong \widetilde{\mathrm{M}}_\eta(K) \oplus \widetilde{\mathrm{M}}_\eta(K)(2k)[4k-1] \in \widetilde{\mathrm{DM}}(K)[\eta^{-1}],$$

$$\mathbf{HS}_{2k+1} := \widetilde{\mathrm{M}}_\eta(V_2(\mathbb{A}^{2k+1})) \cong \widetilde{\mathrm{M}}_\eta(K) \oplus \widetilde{\mathrm{M}}_\eta(K)(4k+1)[8k] \in \widetilde{\mathrm{DM}}(K)[\eta^{-1}]$$

Decompositions of SL_n and GL_n

$$\widetilde{\mathrm{M}}_\eta(\mathrm{SL}_{2i}) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \ldots \otimes \mathbf{HS}_3$$

$$\widetilde{\mathrm{M}}_\eta(\mathrm{SL}_{2i+1}) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \ldots \otimes \mathbf{HS}_3.$$

$$\widetilde{\mathrm{M}}_\eta(\mathrm{GL}_{2i}) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \ldots \otimes \mathbf{HS}_3 \otimes \widetilde{\mathrm{M}}_\eta(\mathbb{G}_m)$$

$$\widetilde{\mathrm{M}}_\eta(\mathrm{GL}_{2i+1}) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \ldots \otimes \mathbf{HS}_3 \otimes \widetilde{\mathrm{M}}_\eta(\mathbb{G}_m)$$

MW-motivic Decomposition on Integral Cases

By combining the result of Stiefel varieties on motivic cohomology, we can expect the following motivic decomposition in $\widetilde{\mathrm{DM}}(K)$. Let $\widetilde{\mathbf{HS}}_{2k+1} := \widetilde{\mathrm{M}}(V_2(\mathbb{A}^{2k+1})) \cong \widetilde{\mathrm{M}}(K) \oplus C_\eta(K)(2k)[4k] \oplus \widetilde{\mathrm{M}}(K)(4k+1)[8k]$

MW-motivic Decomposition on Integral Cases

By combining the result of Stiefel varieties on motivic cohomology, we can expect the following motivic decomposition in $\widetilde{\mathrm{DM}}(K)$. Let $\widetilde{\mathbf{HS}}_{2k+1} := \widetilde{\mathrm{M}}(V_2(\mathbb{A}^{2k+1})) \cong \widetilde{\mathrm{M}}(K) \oplus C_\eta(K)(2k)[4k] \oplus \widetilde{\mathrm{M}}(K)(4k+1)[8k]$

Theorem(WIP)

$$\widetilde{\mathrm{M}}(V_{2j}(\mathbb{A}^{2i})) \cong \widetilde{\mathrm{M}}(\mathbb{A}^{2i}\setminus\{0\})\otimes\widetilde{\mathbf{HS}}_{2i-1}\otimes\widetilde{\mathbf{HS}}_{2i-3}\otimes\ldots\otimes\widetilde{\mathbf{HS}}_{2i-1-2(j-2)}\otimes\widetilde{\mathrm{M}}(\mathbb{A}^{2i+1-2j}\setminus\{0\})$$

$$\widetilde{\mathrm{M}}(V_{2j+1}(\mathbb{A}^{2i})) \cong \widetilde{\mathrm{M}}(\mathbb{A}^{2i}\setminus\{0\}) \otimes \widetilde{\mathbf{HS}}_{2i-1} \otimes \widetilde{\mathbf{HS}}_{2i-3} \otimes \ldots \otimes \widetilde{\mathbf{HS}}_{2i-1-2(j-1)}$$

$$\widetilde{\mathrm{M}}(V_{2j}(\mathbb{A}^{2i+1})) \cong \widetilde{\mathbf{HS}}_{2i+1} \otimes \widetilde{\mathbf{HS}}_{2i-1} \otimes \ldots \otimes \widetilde{\mathbf{HS}}_{2i+1-2(j-1)}$$

$$\widetilde{\mathrm{M}}(V_{2j+1}(\mathbb{A}^{2i+1})) \cong \widetilde{\mathbf{HS}}_{2i+1} \otimes \widetilde{\mathbf{HS}}_{2i-1} \otimes \ldots \otimes \widetilde{\mathbf{HS}}_{2i+1-2(j-1)} \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2i+1-2j}\setminus\{0\}).$$

Thank you

Thank you