Cellular $\mathbb{A}^1$-Homology of Smooth Toric Varieties

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A simplicial complex K on a finite set $[\![m]\!] = \{1, \ldots, m\}$ is defined as a collection of subsets of $[\![m]\!]$ satisfying the following conditions:

1. Any singleton subset $\{v\} \in K$ for all $v \in [\![m]\!]$.
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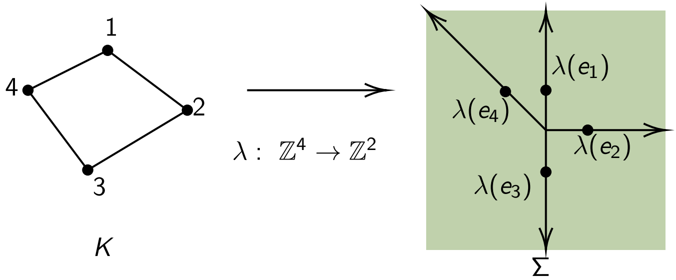

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3. Let $\tau \in \Sigma(k)$ and $\tau \subset \sigma \in \Sigma(n)$ then we have induced isomorphisms for U_τ and Y_τ:

$$\begin{array}{ccccc} U_\sigma & \hookleftarrow & U_\tau & \hookleftarrow & Y_\tau \\ \downarrow{\scriptstyle \varphi_\sigma} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathbb{A}^n & \hookleftarrow & \mathbb{A}^k \times \mathbb{G}_m^{n-k} & \hookleftarrow & \mathbb{G}_m^{n-k} \end{array}$$

Remark

For a cone $\tau \subsetneq \sigma_1, \sigma_2$ that is contained in two distinct maximal cones, we can compare the isomorphisms provided by these maximal cones. The transition morphism

$$g_{12} = \varphi_{\sigma_1} \circ \varphi_{\sigma_2} : \mathbb{A}^k \times \mathbb{G}_m^{n-k} \to \mathbb{A}^k \times \mathbb{G}_m^{n-k}$$

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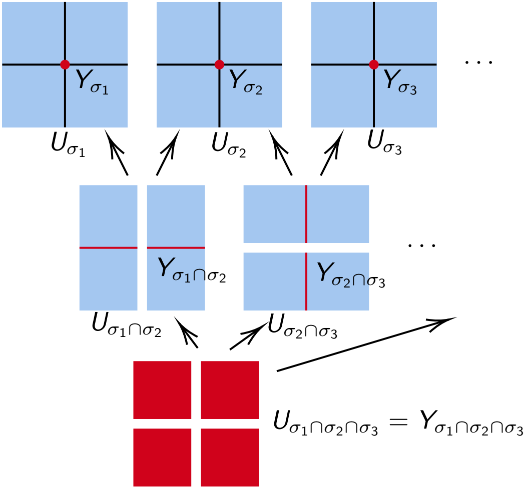
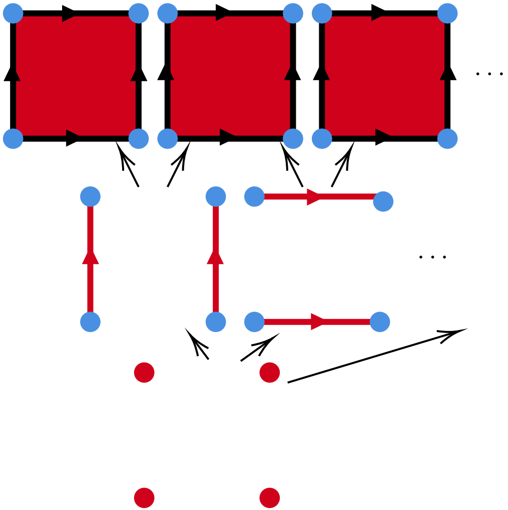
To introduce the concept of cellular $\mathbb{A}^1$-homology, let us initially examine the (topological) cellular structure of the real points $X_\Sigma(\mathbb{R})$.

Cellular Complex of Real Points

The real points $X_\Sigma(\mathbb{R})$ of toric varieties are actually "cubical":



It is also helpful to think about the (Poincare) dual pictures:



Cellular $\mathbb{A}^1$-Homology

$\mathbb{A}^1$-cellular structure

Toric variety X_Σ admit a $\mathbb{A}^1$-cellular structure, defined by a filtration:

$$\mathbb{G}_m^n \cong \Omega_0 \subset \cdots \subset \Omega_n = X_\Sigma$$

where $\Omega_i = \bigcup_{\sigma \in \Sigma(i)} U_\sigma$. This filtration satisfies

$$\Omega_i \setminus \Omega_{i-1} = \coprod_{\sigma \in \Sigma(i)} Y_\sigma \cong \coprod_{\sigma \in \Sigma(i)} \mathbb{G}_m^{n-i},$$

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Then we can choose the orientations for the Thom spaces to obtain the following identification:

$$\iota : \Omega_i/\Omega_{i-1} = \coprod_{\sigma \in \Sigma(i)} \mathrm{Th}(N_{U_\sigma/Y_\sigma}) \xrightarrow{\cong} \coprod_{\sigma \in \Sigma(i)} \mathbb{G}_m^{n-i} \times (\mathbb{A}^i/\mathbb{A}^i \setminus \{0\})$$

On other hand, we define the boundary morphism by composing

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By applying the selected orientations, we can have the following identification:

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Furthermore, we have $\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{G}_m^{n-i} \times (\mathbb{A}^i/\mathbb{A}^i \setminus \{0\})) = \mathbf{H}^{\otimes n-i} \otimes \mathrm{K}_i^{\mathrm{MW}}$, where $\mathbf{H} := \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \cong \mathbb{Z} \oplus \mathrm{K}_1^{\mathrm{MW}}$ (analogous to $\mathbb{Z}[\mathbb{Z}/2] \cong \mathbb{Z} \oplus \mathbb{Z}$).

Thus, we can define the oriented boundary morphism as

$$\widetilde{\partial}_i : \bigoplus_{\sigma \in \Sigma(i)} \mathbf{H}^{\otimes n-i} \otimes \mathrm{K}_i^{\mathrm{MW}} \to \bigoplus_{\sigma \in \Sigma(i-1)} \mathbf{H}^{\otimes n-i+1} \otimes \mathrm{K}_{i-1}^{\mathrm{MW}}$$

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(Oriented) Cellular $\mathbb{A}^1$-Chain Complex and Cellular $\mathbb{A}^1$-Homology

We define the (oriented) cellular $\mathbb{A}^1$-chain complex $C_*^{cell}(X_\Sigma) \in D(Ab_{\mathbb{A}^1}(k))$ as:

$$C_*^{cell}(X_\Sigma) := \left(\bigoplus_{\sigma \in \Sigma(i)} \mathbf{H}^{\otimes n-i} \otimes \mathrm{K}_i^{\mathrm{MW}}, \widetilde{\partial}_i \right)_i$$

whose homology groups are the cellular $\mathbb{A}^1$-homology $\mathbf{H}_*^{cell}(X_\Sigma) \in Ab_{\mathbb{A}^1}(k)$.

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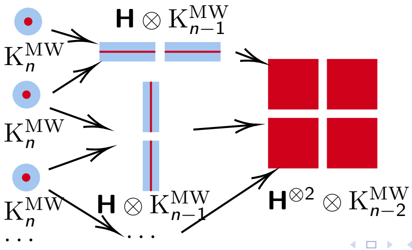

What can we do with cellular $\mathbb{A}^1$-homology?

Proposition [Prop 2.27, Morel-Sawant 23]

For any strictly $\mathbb{A}^1$-invariant sheaf $\mathbf{M} \in Ab_{\mathbb{A}^1}(k)$, we have the isomorphisms

$$H^n_{\mathrm{Nis}}(X_\Sigma, \mathbf{M}) \xrightarrow{\cong} \mathrm{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C^{cell}_*(X_\Sigma), \mathbf{M}[n])$$

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We can use cellular $\mathbb{A}^1$-chain complex to compute MW-motive $\widetilde{\mathrm{M}}(X_\Sigma) \in \widetilde{\mathrm{DM}}(k)$, more precisely

$$\widetilde{\mathrm{M}}(X_\Sigma)_+ := \left(\bigoplus_{\sigma \in \Sigma(i)} \widetilde{\mathrm{M}}(\mathbb{G}_m)_+^{\otimes n-i} \otimes \widetilde{\mathbb{Z}}(i)[i], \widetilde{\partial}_i \right)_i$$

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Question

How do we determine the boundary morphism $\widetilde{\partial}_i$?

Cubical Cells

In order to make the explicit computation, we need to define a "basis" for $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m^I) \cong \mathbf{H}^{\otimes I}$, which has 2^I summands.

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$$e = \prod_{i \in \tau_e^1} \{x_i = 1\} \prod_{i \in \tau_e^{\odot}} \{x_i \neq 0\} \subset \mathbb{G}_m^l$$

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Furthermore, it is easy to see that

$$\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m^l) \cong \bigoplus_{e \subset \mathbb{G}_m^l \text{ cubical}} [e]\mathrm{K}_{t_e}^{\mathrm{MW}}$$

Similarly, we can define the oriented cubical cells $[e, \theta] : \mathrm{K}_{t_e+i}^{\mathrm{MW}} \hookrightarrow C_i^{cell}(X_\Sigma)$ and form a basis.

Example: Cubical Cellular Structure of $\mathbb{A}^n$

Example

$\mathbb{A}^n$ can be regarded as a toric variety with the fan $(2^{[\![n]\!]}, \mathrm{Id})$.

$$C_*^{cell}(\mathbb{A}^n) := \left(\bigoplus_{\sigma \subset [\![n]\!],\ |\sigma| = i} \mathbf{Z}_{\mathbb{A}^1}[Y_\sigma] \otimes \mathrm{K}_i^{\mathrm{MW}}, \partial_i \right)_i$$

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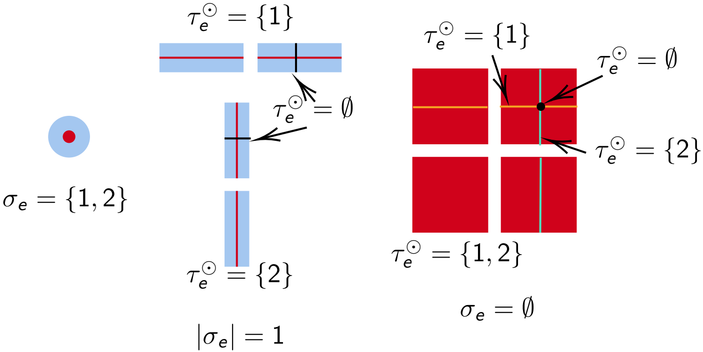
Cubical Cells of $\mathbb{A}^n$

Given a partition of $\tau_e^1 \sqcup \tau_e^\odot \sqcup \sigma_e = [\![n]\!]$, we define the cubical cell as

$$e = \prod_{i \in \tau_e^1} \{x_i = 1\} \prod_{i \in \tau_e^\odot} \{x_i \neq 0\} \prod_{i \in \sigma_e} \{x_i = 0\} \subset Y_{\sigma_e} \subset \mathbb{A}^n$$

This provides a basis $C^{cell}_i(\mathbb{A}^n) = \bigoplus_{\sigma_e \subset [\![n]\!],\ |\sigma_e| = i} [e] \mathrm{K}^{\mathrm{MW}}_{t_e + i}$, where $\partial[e] = \sum_{j \in \sigma_e} \pm \epsilon^?[\partial_j e]$ ($\partial_j e$ signifies moving $j \in \sigma_e$ to $\tau_e^\odot$).


$\sigma_e = \{1,2\}$
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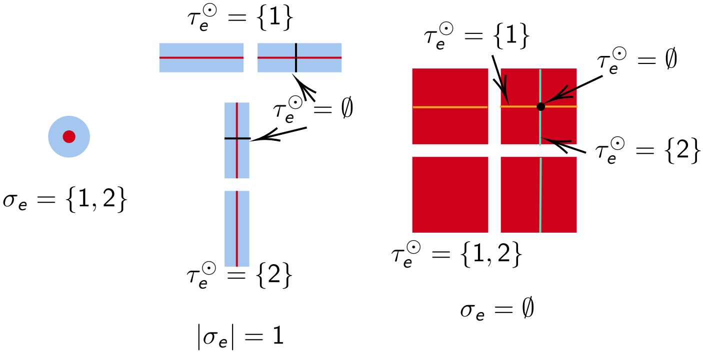

The intuition is that σ_e determines the dual dimension of the cell $[e]$, while $\tau_e^{\odot}$ indicates the number of connected components it contains (i.e., 2^{t_e}).
It is worth noting that, although $C_*^{cell}(\mathbb{A}^n) \cong \mathbb{Z}$, this cellular structure serves as the fundamental block (cube) for toric varieties.

Moment-Angle Complexes and Toric Quotient

The boundary morphisms can become quite complex in higher dimensions, instead, we can use the fact that X_Σ is a toric quotient.

Example

Consider the projective space:

$$\mathbb{G}_m \circlearrowright \mathbb{A}^n \setminus \{0\} \longrightarrow \mathbb{A}^n$$

$$\downarrow$$

$$\mathbb{P}^{n-1} \cong \mathbb{A}^n \setminus \{0\}/\mathbb{G}_m$$

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In general, for the simplicial complex K over $[\![m]\!]$, we can define the moment-angle complex $\mathbb{A}\mathcal{Z}_K \subset \mathbb{A}^m$ as:

Moment-Angle Complex

$$\mathbb{A}\mathcal{Z}_K := \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in \mathbb{A}^m \mid x_i \neq 0 \text{ if } i \notin \sigma\}$$

The upshot is that $C_i^{cell}(\mathbb{A}\mathcal{Z}_K) = \bigoplus_{\sigma_e \in K,\ |\sigma_e|=i}[e]\mathrm{K}^{\mathrm{MW}}_{t_e+i}$, and $C_*^{cell}(\mathbb{A}\mathcal{Z}_K)$ is a subcomplex of $C_*^{cell}(\mathbb{A}^m)$, making the boundary morphism straightforward.

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Now for a smooth fan $\Sigma = (K, \lambda : \mathbb{Z}^m \twoheadrightarrow \mathbb{Z}^n)$, consider the morphism $\exp\lambda : \mathbb{G}_m^m \twoheadrightarrow \mathbb{G}_m^n$ induced by λ, and let $G := \mathrm{Ker}(\exp\lambda) \subset \mathbb{G}_m^m$ be its kernel.

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Homogeneous Coordinate

G acts freely on $\mathbb{A}\mathcal{Z}_K$, and we have

$$\begin{array}{ccc} G \circlearrowright \mathbb{A}\mathcal{Z}_K & \longrightarrow & \mathbb{A}^m \\ \downarrow p & & \\ X_\Sigma \cong \mathbb{A}\mathcal{Z}_K / G & & \end{array}$$

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This induced a morphism of complexes

$$\begin{aligned} p_* : \quad C_*^{cell}(\mathbb{A}\mathcal{Z}_K) &\to C_*^{cell}(X_\Sigma) \\ [e] &\mapsto [p(e)] \end{aligned}$$

such that for any group section $g : \mathbb{G}_m^{t_e} \to G$, $p_*[e] = p_* g_*[e]$.

Toric Action on Cubical Cells

The toric group $\mathbb{G}_m^n$ naturally acts on the affine space $\mathbb{A}^n$. To understand its action on a cubical cell $e : \mathbb{G}_m^{t_e} \to \mathbb{A}^n$, consider a group section $g : \mathbb{G}_m^{t_e} \to \mathbb{G}_m^n$. The image $g_*[e]$ corresponds to the cell $g \cdot e : \mathbb{G}_m^{t_e} \to \mathbb{A}^n$, which may not be a cubical cell.

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To represent $g_*[e]$ in terms of cubical cells, notice that g can be represented by a $t_e \times n$ matrix over $\mathbb{Z}$ as $\{r_{ij}\}$, and defining $r'_{ij} = r_{ij} + \delta_{ij}$. Additionally, let $\chi(i) = 0$ for even i and 1 for odd i.

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Proposition

Let $\omega_0 \subset \tau_e = \tau_e^1 \sqcup \tau_e^{\odot}$ be the subset such that $j \in \omega_0$ if $\forall i \in \tau_e^{\odot}, r'_{ij} = 0$. If $t_e > |\tau_e| - |\omega_0|$, then:

$$g_*[e] = \sum_{\omega \subset \tau_e \setminus \omega_0} \eta^{t_e - |\omega|} \sum_{\pi \subset \omega} (-1)^{|\omega| - |\pi|} \prod_{i \in \tau_e^{\odot}} \chi\left(\sum_{j \in \pi \sqcup \sigma_e} r'_{ij} \right) [e_\omega]$$

Here we have the morphism $\eta : \mathrm{K}_i^{\mathrm{MW}} \to \mathrm{K}_{i-1}^{\mathrm{MW}}$ which is analogous of multiplying by 2. And $[e_\omega]$ represents the cubical cell where $\tau^{\odot} = \omega$.

Canonical Cells

To further simplify the computation, we introduce a complex $C_*^{can}(\mathbb{A}\mathcal{Z}_K)$ consisting of canonical cells.

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$$\begin{array}{ccc} C_i^{can}(\mathbb{A}\mathcal{Z}_K) & \subset & C_i^{cell}(\mathbb{A}\mathcal{Z}_K) \\ & \nwarrow^{\cong} & \downarrow p_* \\ & & C_i^{cell}(X_\Sigma) \end{array}$$

And for any cubical $[e] \in C_i^{cell}(\mathbb{A}\mathcal{Z}_K)$, there exists a unique group section $T_e : \mathbb{G}_m^{t_e} \to G$, such that $T_*[e] := T_{e*}[e] \in C_i^{can}(\mathbb{A}\mathcal{Z}_K)$.

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The compatible boundary morphism can then be defined as:

$$\partial^{can}[e] = \sum_{j \in \sigma_e} \pm \epsilon^? T_*([\partial_j e]).$$

And p_* induces an isomorphism of complexes $p_* : C_*^{can}(\mathbb{A}\mathcal{Z}_K) \xrightarrow{\cong} C_*^{cell}(X_\Sigma)$.

Shellable Simplicial Complex

We can further simplify $C_*^{can}(\mathbb{A}\mathcal{Z}_K)$ by considering the restriction complex $\overline{C}_*^{can}(\mathbb{A}\mathcal{Z}_K) \subset C_*^{can}(\mathbb{A}\mathcal{Z}_K)$, which acts as a retraction, leading to an isomorphism $\overline{C}_*^{can}(\mathbb{A}\mathcal{Z}_K) \cong C_*^{can}(\mathbb{A}\mathcal{Z}_K)$.

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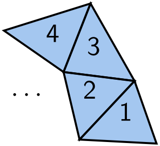
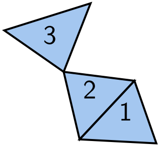
Let's consider the sheallable cases. A simplicial complex K is shellable if it admits a shelling, i.e. an ordering $\{\sigma_2, \ldots, \sigma_s\} = K(n)$ of its facets such that they have nice intersections.



Shellable



Non-shellable

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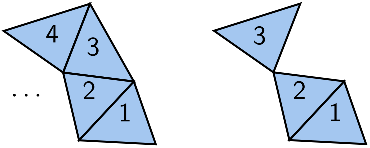

Shellable   Non-shellable

The shellable simplicial complex have the important property that for $\sigma_i \in K(n)$, here exists a unique subset $r(\sigma_i) \subset \sigma_i$ that is minimal among all subsets $\tau \subseteq \sigma_i$, where $\tau \nsubseteq \sigma_j$ for all $j < i$.

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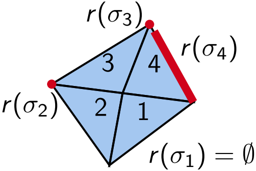

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For a shellable fan $\Sigma = (K, \lambda)$, we have the restriction complex $\overline{C}_*^{can}(\mathbb{A}\mathcal{Z}_K)$ where on each degree it only depends on K:

$$\overline{C}_i^{can}(\mathbb{A}\mathcal{Z}_K) = \bigoplus_{\sigma \in K(n),\ |r(\sigma)|=i} [e_\emptyset^{r(\sigma)}]\mathrm{K}_i^{\mathrm{MW}}$$

and $[e_\emptyset^{r(\sigma)}]$ means $\sigma_e = r(\sigma)$ and $\tau_e^{\odot} = \emptyset$.

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and $[e_\emptyset^{r(\sigma)}]$ means $\sigma_e = r(\sigma)$ and $\tau_e^{\odot} = \emptyset$.
While boundary morphism $\overline{\partial}^{can}$ also relies on λ.

Proposition

$$\overline{\partial}^{can}[e_\emptyset^{r(\sigma)}] = \sum_{j \in r(\sigma)} \pm w_j \eta [e_\emptyset^{r(\sigma)\setminus j}]$$

where $w_j = 0$ or 1 depends on the fan $\Sigma = (K, \lambda)$.

We can in fact explicitly calculus the homology using some combinatorial ways.

Low Dimensional Cases

Given that K is the boundary complex of a simple n-polytope (e.g., when X_Σ is projective), we can show that for dimension $n \leq 4$, the torsion part of $\mathbf{H}_i^{cell}(X_\Sigma)$ consist only of η-torsion elements.
Recall that $\mathrm{K}_i^{\mathrm{MW}}/\eta \cong \mathrm{K}_i^{\mathrm{M}}$ and ${}_\eta\mathrm{K}_i^{\mathrm{MW}} \cong 2\mathrm{K}_i^{\mathrm{M}}$. Let $b_i = \mathrm{rk}(G_{i,free}^{\lambda})$ be some Betti numbers can be computed from the fan Σ, and X_Σ^n denotes the toric variety of dimension n.

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$\mathbf{H}_i^{cell}(X_\Sigma^2)$	Orientable	Non-orientable
$i=0$	$\mathbb{Z}$	$\mathbb{Z}$
$i=1$	$(\mathrm{K}_1^{\mathrm{MW}})^{m-2}$	$(\mathrm{K}_1^{\mathrm{MW}})^{m-3} \oplus \mathrm{K}_2^{\mathrm{M}}$
$i=2$	$\mathrm{K}_2^{\mathrm{MW}}$	$2\mathrm{K}_2^{\mathrm{M}}$

$\mathbf{H}_i^{cell}(X_\Sigma^3)$	Orientable	Non-orientable
$i=0$	$\mathbb{Z}$	$\mathbb{Z}$
$i=1$	$(\mathrm{K}_1^{\mathrm{MW}})^{b_0} \oplus (\mathrm{K}_1^{\mathrm{M}})^{m-3-b_0}$	$(\mathrm{K}_1^{\mathrm{MW}})^{b_0} \oplus (\mathrm{K}_1^{\mathrm{M}})^{m-3-b_0}$
$i=2$	$(\mathrm{K}_2^{\mathrm{MW}})^{b_0} \oplus (2\mathrm{K}_2^{\mathrm{M}})^{m-3-b_0}$	$(\mathrm{K}_2^{\mathrm{MW}})^{b_0-1} \oplus \mathrm{K}_2^{\mathrm{M}} \oplus (2\mathrm{K}_2^{\mathrm{M}})^{m-3-b_0}$
$i=3$	$\mathrm{K}_3^{\mathrm{MW}}$	$2\mathrm{K}_3^{\mathrm{M}}$

$\mathbf{H}_i^{cell}(X_\Sigma^4)$	Orientable
$i=0$	$\mathbb{Z}$
$i=1$	$(\mathrm{K}_1^{\mathrm{MW}})^{b_0} \oplus (\mathrm{K}_1^{\mathrm{M}})^{m-4-b_0}$
$i=2$	$(\mathrm{K}_2^{\mathrm{MW}})^{b_1} \oplus (\mathrm{K}_2^{\mathrm{M}})^{m-4-b_0} \oplus (2\mathrm{K}_2^{\mathrm{M}})^{m-4-b_0}$
$i=3$	$(\mathrm{K}_3^{\mathrm{MW}})^{b_0} \oplus (2\mathrm{K}_3^{\mathrm{M}})^{m-4-b_0}$
$i=4$	$\mathrm{K}_4^{\mathrm{MW}}$
$\mathbf{H}_i^{cell}(X_\Sigma^4)$	Non-orientable
$i=0$	$\mathbb{Z}$
$i=1$	$(\mathrm{K}_1^{\mathrm{MW}})^{b_0} \oplus (\mathrm{K}_1^{\mathrm{M}})^{m-4-b_0}$
$i=2$	$(\mathrm{K}_2^{\mathrm{MW}})^{b_1} \oplus (\mathrm{K}_2^{\mathrm{M}})^{m-5-b_2} \oplus (2\mathrm{K}_2^{\mathrm{M}})^{m-4-b_0}$
$i=3$	$(\mathrm{K}_3^{\mathrm{MW}})^{b_2} \oplus \mathrm{K}_3^{\mathrm{M}} \oplus (2\mathrm{K}_3^{\mathrm{M}})^{m-5-b_2}$
$i=4$	$2\mathrm{K}_4^{\mathrm{M}}$

We can observe the Poincare duality for complex point $X_\Sigma(\mathbb{C})$ by replacing K^{MW} and K^{M} with $\mathbb{Z}[1]$, and for real points $X_\Sigma(\mathbb{R})$ by removing K^{M} and replacing K^{MW} with $\mathbb{Z}$.

Motivic Decomposition

Corollary

In the category $\widetilde{\mathrm{DM}}(k)$, for a smooth pure shellable toric variety X_Σ, we have the following MW-motivic decomposition:

$$\widetilde{\mathrm{M}}(X_\Sigma) \cong \bigoplus_{I \in \mathbb{N}} \bigoplus_{\sigma \in B(I)} \widetilde{\mathbb{Z}} /\!\!/ I\eta(|r(\sigma)|)[2|r(\sigma)|],$$

where $B(I) \subset K(n)$ are subsets that can be derived from the fan $\Sigma = (K, \lambda)$.

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$$\widetilde{\mathrm{M}}(X_\Sigma) \cong \bigoplus_{l \in \mathbb{N}} \bigoplus_{\sigma \in B(l)} \widetilde{\mathbb{Z}} /\!\!/ l\eta(|r(\sigma)|)[2|r(\sigma)|],$$

where $B(l) \subset K(n)$ are subsets that can be derived from the fan $\Sigma = (K, \lambda)$.

If we pass to the derived motivic category $\mathrm{DM}(k)$, in which $\eta = 0$, then we obtain this trivial corollary (as an analogue of $\mathbb{Z}/2$ coefficient):

Corollary

In the category $\mathrm{DM}(k)$, for a smooth pure shellable toric variety X_Σ, we obtain the following motivic decomposition:

$$\mathrm{M}(X_\Sigma) \cong \bigoplus_{\sigma \in K(n)} \mathbb{Z}(|r(\sigma)|)[2|r(\sigma)|]$$

More General Cases

Our results do not hold exactly when the pure or shellable conditions are removed. The problem arises from the non-algebro-geometric components (i.e., the summands $\mathbb{Z}(q)[p]$ with $2q > p$) it has. However, these components vanish when considering only the Chow group, providing an additive basis for the Chow group.

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Proposition

For a smooth toric variety X_Σ, consider an order on $K(n) = \{\sigma_1, \ldots, \sigma_g\}$. Define the sets

$$\min(\sigma_i) = \{\tau \subset \sigma_i \mid \tau \text{ is minimal for } \tau \not\subset \sigma_j \text{ for all } j < i\}.$$

We then have the following decomposition of the Chow group:

$$\mathrm{CH}^*(X_\Sigma) \cong \bigoplus_{\sigma \in K_{\max}} \bigoplus_{\tau \in \min(\sigma)} \mathbb{Z}[e^\tau]$$

The generators are given by $[e^\tau] \in \mathrm{CH}^{|\tau|}(X_\Sigma)$.

Thank you

Questions

What is $B(l)$?

Given a mod-2 linear function $\kappa : \mathbb{Z}^n \to \mathbb{Z}_2^n \to \mathbb{Z}_2$, we define the row set $\omega_\kappa \subset [\![m]\!]$ as the subset $\{j \in [\![m]\!] \mid \kappa\lambda(v_j) \equiv 1 \mod 2\}$, where v_j are the basis vectors of $\mathbb{Z}^m$. Let $\mathrm{row}\lambda \subset 2^{[\![m]\!]}$ denote the set of all row sets; thus, we can observe that $|\mathrm{row}\lambda| = 2^n$.

For $\lambda : \mathbb{Z}^m \to \mathbb{Z}^n$, let K_ω represent a specific subcomplex of K formed by intersecting with $\omega \in \mathrm{row}\lambda$. We define $G_i^\lambda = \bigoplus_{\omega \in \mathrm{row}\lambda} \widetilde{H}_i(|K_\omega|)$ as the direct sum of the reduced homology groups. The basis $B(0) \subset K_{\max}$ forms the free part of $G^\lambda_{*,free}$, while for $l > 1$, the basis $B(l) \subset K_{\max}$ corresponds to the l-torsion part $G^\lambda_{*,l-tor}$.